Test 3, Linear Algebra

Dr. Adam Graham-Squire, Fall 2017

34 : 37

Name: Key

I pledge that I have neither given nor received any unauthorized assistance on this exam.

(signature)

DIRECTIONS

1. Don't panic.
2. Show all of your work. A correct answer with insufficient work will lose points.
3. Read each question carefully and **make sure you answer the question that is asked**. If the question asks for an explanation, make sure you give one.
4. Clearly indicate your answer by putting a box around it.
5. Calculators are allowed on this exam, though they are not necessary.
6. Make sure you sign the pledge.
7. The first 10 questions are required, and I will drop your lowest score of the last three questions.
8. Number of questions = 13. Total Points = 60.

ID Number: ______________________

$(1-2t+t$

1. (6 points) Consider the following set of three polynomials in $\mathbb{P}_3$:

$$\{1+t^2+t^3, 1-t, (1-t)^2\}$$

$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

Answer the following, and justify your answer:

(a) Is the set linearly independent?

(b) Does the set span $\mathbb{P}_3$?

(c) Is the set a basis for $\mathbb{P}_3$? If not, what could you do to it to make it a basis?

(a)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Yes! b/c all columns have a pivot

(b) No, b/c 3 pivots and 4 rows, so 1 row has no pivot

(c) Not a basis! Need to add one more vector that would be linearly indep. from the rest.

2. (3 points) Find the characteristic polynomial and eigenvalues for the matrix $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

The characteristic polynomial should be written in factored form. Show your work for how you got your answers, but you can use a calculator or computer to check your work.

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & -2 \\ 1 & 3-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(3-\lambda)(3-\lambda) + 0 + 0 - (2)(2-\lambda) - 0 - 0$$

$$= \boxed{(2-\lambda)(3-\lambda)(3-\lambda) \text{ is char. polyn.}}$$

$\Rightarrow$ $\boxed{\text{eigenvalues are } 2, 3, \text{ and } 3}$

3. (6 points) Is the matrix $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ from problem 2 diagonalizable? If so, find D and P. If not, explain why it is not diagonalizable. Show your work for how you got your answers, but you can use a calculator or computer to check your work.

✓ $\lambda = 3$ has multiplicity two, need to check its eigenspace.

✓ $A - 3I = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has two free variables $\Rightarrow$ dim (eigenspace for $\lambda = 3$) $= 2$

$\Rightarrow$ diagonalizable!

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \checkmark$$

Need eigenvectors for $\lambda = 2$, $\lambda = 3$.

for $\lambda = 3$ get $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ $x_1 + 2x_3 = 0$, $x_2 = x_2$, $x_3 = x_3$ $\Rightarrow x = \begin{bmatrix} -2x_3 \\ x_2 \\ x_3 \end{bmatrix}$

✓ $= x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

for $\lambda = 2$ get $A - 2I = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ ✓ $x_1 + x_2 = 0$, $x_3 = 0$, x_2 is free $\Rightarrow \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix}$

$\Rightarrow x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

No Calculator section Name: Key

4. (3 points) A is a 6×6 matrix with 4 distinct eigenvalues. One eigenspace is two-dimensional. Is A diagonalizable? Your answer should be Yes, No, or Can't Say, and you should explain your reasoning.

✓ Cant say for sure. ~~Bash~~ With 4 distinct eigenvalues, could have multiplicities 1, 1, 1, 3 or 1, 1, 2, 2.

- If 1, 1, 1, 3 and one eigenspace of dim = 2,

✓✓ then <u>not</u> diagonalizable b/c needs dim = 3.

- If 1, 1, 2, 2, then we need to find dim of eigenspace for other mult. 2 λ. If dim = 2 then <u>yes</u> diagonalizable. If dim = 1, then <u>No</u>.

5. (~~4~~ 3 points) True or False: If true, briefly explain why. If false, explain why or give a counterexample.

(i) Let A be an $m \times n$ matrix. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b}$, then Col A is all of $\mathbb{R}^m$.

✓ False! If $b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $Ax = b$ is consistent, but $Ax = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 2 \end{bmatrix}$ is not.

✓

(ii) To find the eigenvalues of A, reduce A to echelon form.

✓ False! Need to do $A - \lambda I$ and calculate char. polynomial! ✓

(iii) A change of coordinates matrix is always invertible.

✓ True! The columns for a COC matrix form a basis, thus are lin. indep and span, so by IMT they are invertible. ✓

(iv) If $\mathbf{f}$ is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $\mathbf{f}(t) = 0$ for some t, then $\mathbf{f}$ is the zero vector in V.

~~False! Need to have $f(t) = 0$ for all values of t.~~

(v) A diagonalizable matrix is always invertible.

✓ False! Can have $\lambda = 0$ be an eigenvalue $\Rightarrow$ Not invertible.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is diagonalizable but not invertible.

✓

6. (4 points) Multiple Choice Questions. You do *not* need to show any work to receive full credit for these problems (although showing work can get you partial credit if your answer is wrong).

(a) The vector space of polynomials of degree 3 or less is $\mathbb{P}_3$, it has dimension

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

$1+t+t^2+t^3$

(b) The subspace $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$ has dimension

b/c lin. dependent.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

$\Rightarrow$ 2 pivots

(c) If the column space of a 3×6 matrix A is 2-dimensional, what is the dimension of the null space of A?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5 (g) 6 (h) 7

$$\begin{array}{cc} & 6 \\ 3 & \begin{bmatrix} 1 & 1 & * & * & * & * \end{bmatrix} \end{array}$$

$\hookrightarrow$ 4 free

rank A + dim (Nul A) = 6

$2 + \underline{\quad} = 6$

(d) If A is a 2×4 matrix, what is the largest possible rank of A?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5 (g) 6

$$\begin{array}{cc} & 4 \\ 2 & \begin{bmatrix} \quad \end{bmatrix} \end{array}$$

greatest rank = 2 = # of rows.

(e) If A is a 3×2 matrix, what is the smallest possible dimension of the nullspace of A?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5 (g) 6

$$\begin{array}{cc} & 2 \\ 3 & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$\Rightarrow$ could have rank = 2 $\Rightarrow$ Nul $A = \vec{0}$ only

$\Rightarrow$ dim(Nul A) = 0

7. (6 points) Consider the following two systems of equations:

$$\begin{aligned} 5x_1 + \ x_2 - \ 3x_3 &= 2 \\ -9x_1 + 2x_2 + 5x_3 &= 3 \\ 4x_1 + \ x_2 - \ 6x_3 &= 9 \end{aligned} \quad \text{and} \quad \begin{aligned} 5x_1 + \ x_2 - \ 3x_3 &= -6 \\ -9x_1 + 2x_2 + 5x_3 &= -9 \\ 4x_1 + \ x_2 - \ 6x_3 &= -27 \end{aligned}$$

Suppose you know that the first system has a solution. Use this fact to explain why the second system also has a solution without making any row operations.

Let $A = \begin{bmatrix} 5 & 1 & -3 \\ -9 & 2 & 5 \\ 4 & 1 & -6 \end{bmatrix}$

Then $Ax = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}$ has a solution $\vec{p}$

Consider $-3\vec{p}$. Then =

$A(-3\vec{p}) = -3(A\vec{p}) = -3\begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \\ -27 \end{bmatrix}$

So $-3\vec{p}$ is a solution to the second system of equations.

Better answer: • 1st system has solution $\Rightarrow \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}$ is in Col A

• Col A a subspace $\Rightarrow -3\begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \\ -27 \end{bmatrix}$ is also in Col

$\Rightarrow$ second system has a solution.

8. (6 points) $M_{2\times 3}$ is the vector space of all 2×3 matrices with normal addition and scalar multiplication. Determine if the set H of all matrices of the form $\begin{bmatrix} a & b & 0 \\ 0 & 0 & f \end{bmatrix}$, where a, b, f are real numbers, is a subspace of $M_{2\times 3}$.

- Does H have $\vec{0}$? Yes, let $a, b, f = 0$ ✓

- ~~Do~~ Is $h_1 + h_2$ in H? Check

$$\begin{bmatrix} a_1 & b_1 & 0 \\ 0 & 0 & f_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & 0 \\ 0 & 0 & f_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 & 0 \\ 0 & 0 & f_1+f_2 \end{bmatrix}$$ ✓ ✓

is in H, so yes. ✓

- Is ch in H? Yes, b/c $c\begin{bmatrix} a & b & 0 \\ 0 & 0 & f \end{bmatrix}$ ✓ ✓

$$= \begin{bmatrix} ca & cb & 0 \\ 0 & 0 & cf \end{bmatrix}$$

is in H. ✓

Yes, it is a subspace.

✓

+1.5 for naming 3 things, saying Yes.

A

9. (4 points) Is $\lambda = -1$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$? If yes, find the corresponding eigenspace.

$A - (-1I) = A + I$

$= \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

reduces to $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

So yes, $\lambda = -1$ is an eigenvalue!

do $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow 2x_1 + x_2 = 0$, x_2 is free $\Rightarrow \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} \Rightarrow x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

eigenspace is all scala multiples of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

~~10~~ 9

10. (6 points) Let $A = \begin{bmatrix} 1 & -2 & 3 & 5 & 8 \\ 2 & -4 & 6 & 15 & 21 \\ 3 & -6 & 9 & 15 & 22 \\ -1 & 2 & -3 & 0 & -1 \end{bmatrix}$. An echelon form for A is $\begin{bmatrix} 1 & -2 & 3 & 5 & 8 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

and its reduced echelon form is $\begin{bmatrix} 1 & -2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer the following (Show work if necessary):

(a) Find the dimension of and a basis for Nul A.

(b) Find the dimension of and a basis for Col A.

(c) Find the dimension of and a basis for Row A.

(d) Find the rank of A.

✓✓ 0.5

b/c 2 free variable

(a) dim (Nul A) = 2, basis is

$x_1 - 2x_2 + 3x_3 = 0$
x_2, x_3 free
$x_4 = 0$
$x_5 = 0$

$\Rightarrow \begin{bmatrix} 2x_2 - 3x_3 \\ x_2 \\ x_3 \\ 0 \\ 0 \end{bmatrix}$

basis is $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

1.5 (b) dim (Col A) = 3 b/c 5 - 2 = 3

basis is 1st, 4th, 5th column $\Rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 15 \\ 15 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 21 \\ 22 \\ -1 \end{bmatrix}$

1.5 (c) dim (Row A) = 3 b/c dim(Row A) = dim (Col A)

basis is $[1 \ -2 \ 3 \ 0 \ 0], [0 \ 0 \ 0 \ 1 \ 0], [0 \ 0 \ 0 \ 0 \ 1]$

0.5 (d) rank A = dim (Col A) = 3

You only need to do two of these last three questions, but you can do all of them and I will take your 2 highest scores.

11. (6 points) Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. [Hint: Suppose a nonzero $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$.]

$A\vec{x} = \lambda\vec{x}$ for some x

A invertible $\Rightarrow$ $A^{-1}A\vec{x} = A^{-1}(\lambda\vec{x})$

$\vec{x} = \lambda A^{-1}\vec{x}$ ← know $\lambda^{-1} \neq 0$ b/c A is invertible

$\Rightarrow \lambda^{-1}\vec{x} = \lambda^{-1}\lambda A^{-1}\vec{x}$

$\lambda^{-1}\vec{x} = A^{-1}\vec{x}$

$\Rightarrow$ $\boxed{\lambda^{-1} \text{ is eigenvalue of } A^{-1}}$

12. (6 points) Let C be the vector space of all continuous functions on the interval $[0,1]$. Define $T : C \to C$ to be the transformation as follows: for the function f, let $T(f)$ be the *antiderivative* F of f such that $F(0) = 0$ (so, for example, $T(x^2) = \frac{x^3}{3}$).

(a) Show that T is a linear transformation

(b) Describe the kernel of T (that is, describe all functions such that $T(f) = \mathbf{0}$).

(a) $T(f_1 + f_2) = \int f_1 + f_2 = \int f_1 + \int f_2$

$= T(f_1) + T(f_2)$

Note no issue with constant of integ. b/c $F(0)=0$

$T(cf_1) = \int c f_1 = c\int f_1 = c\,T(f_1)$

(b) Kernel of T is all functions f such that

$T(f) = 0$ 6 But $T(f) = F + 0$

function + constant of integration

and the only function whose <u>antiderivative</u> is zero,

is $f(x) = 0$ (that is, the zero function)

so $\boxed{\mathrm{Ker}(T) = \{0\}}$

13. (6 points) Let $T : V \to W$ be a linear transformation. Let H be a nonzero subspace of V, and let $T(H)$ be the set of images of vectors in H. Then $T(H)$ is a subspace of W, since the range of a linear transformation is always a subspace. Prove that $\dim T(H) \leq \dim H$. [Hint: Every vector in $T(H)$ has the form $T(\mathbf{x})$ for some $\mathbf{x}$ in H. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V, and write $\mathbf{x}$ as a linear combination of the $\mathbf{v}_i$. Apply T to both sides and use this to argue that a basis for $T(H)$ can have at most n vectors.]

Claim: $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ spans $T(H)$.

Let $\vec{q}$ be in $T(H)$. Then $\vec{q} = T(\vec{x})$ for some x in V.

Since $v_1, \ldots, v_n$ is a basis for V, know

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

$$\Rightarrow \vec{q} = T(\vec{x}) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

so $\vec{q}$ is a linear combination of $\{T(v_1), \ldots, T(v_n)\}$

and thus $\{T(v_1), \ldots, T(v_n)\}$ spans $T(H)$. Since it is a spanning set, the basis for $T(H)$ can have at most n vectors $\Rightarrow$ ~~dim T(H) ≤~~ n

$$\dim T(H) \leq n = \dim H$$

so $\dim T(H) \leq H$ ✓

Extra Credit (2 points): Let S be the set of all 2×2 matrices A such that A has an eigenvalue of zero. Is S a subspace of $M_{2\times 2}$?

No! $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

No e.values of zero!

eigvalue of 0

evalue of 0